p-adic and random p-adic electrostatics

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Topologically, the p-adic integers are the set of paths of an infinite rooted directed tree where each node has p children. The elementary neighborhoods are in correspondence with the nodes and consist of all paths which descend from a given node. The natural (Haar) measure on Z_p can be thought of as 'inheritance measure', where each neighborhood/node divides its measure equally between its immediate children (and the measure associated to the root is 1), and the p-adic distance between paths is equal to the inheritance of their least common ancestor. This notion of measure and distance can be generalized to the pro finite completions of many other infinite rooted trees.

p-adic electrostatics begins with a system of p-adic particles whose interaction energy is proportional to the p-adic distance between them. This system is put in contact with a heat bath at inverse temperature \beta and we study the fluctuations of the locations of the particles as \beta and the particle number N vary. This is encoded in the (canonical and grand canonical) partition function for the physical system. The canonical partition function arises when we fix our system to have N particles and can be realized as an Iguana local zeta function in \beta (the integrand is the \beta-power of the Vandermonde determinant in N variables and the domain of integration is Z_p^N). The grand canonical partition function is the generating function for the canonical partition functions, with the generating variable t representing the cost of adding an additional particle to the system. This latter generating function satisfies a functional equation that allows us to “solve” for the Igusa zeta/canonical partition function via a quadratic recursion in N. This functional equation arises from the “fractal” nature of the p-adics, and verifies that the canonical partition function is a rational function in p^{-\beta} (as guaranteed by Igusa).

If we generalize our rooted tree to allow each child node to independently have an identical, random, positive number of children (specified by the integer-value random variable Q), then we arrive at an infinite branching process, and the pro-finite completion of such a random tree can be thought of a random analog of the p-adics. The ensemble of all such random profinite completions, determined by Q is called the (random) Q-adics, and Q induces a probability measure on this ensemble. Being random, and essentially structureless, the electrostatics (as determined by the partition functions specified by the generalized measure and distance functions) on any random is difficult to determine. However, the averaged partition functions (averaged over the Q-adics) satisfies a functional equation and quadratic recurrences such as arise in the p-adic case. The averaged canonical partition function is a generalization of an Igusa zeta function and, in this instance at least, shares similar properties. In particular the averaged canonical partition function is a rational function in (p^{-\beta}: Q has a non-zero probability of taking the value p} and has a functional equation and quadratic recurrence related to that for each p-adic canonical partition function for which Q takes p with non-zero probability.